Transformation

Computer Graphics

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Outline

- Overview (world transformation)
- Transformation
- OpenGL implementation

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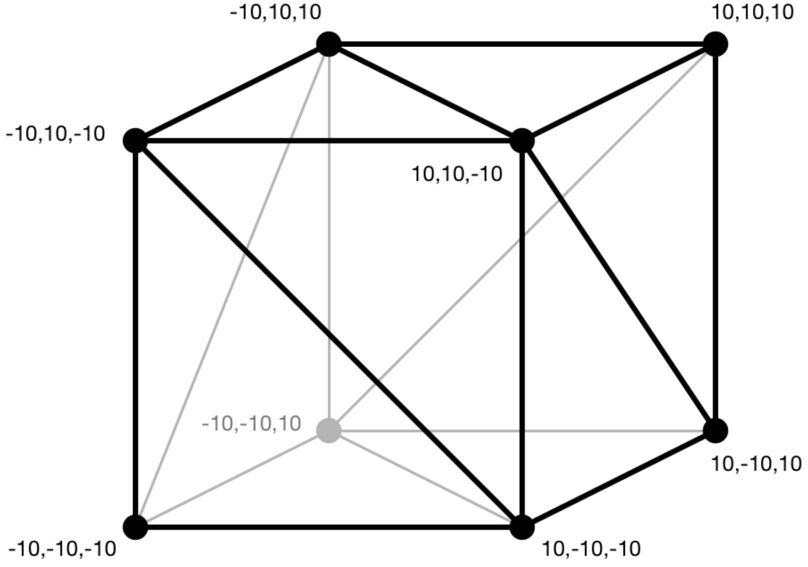
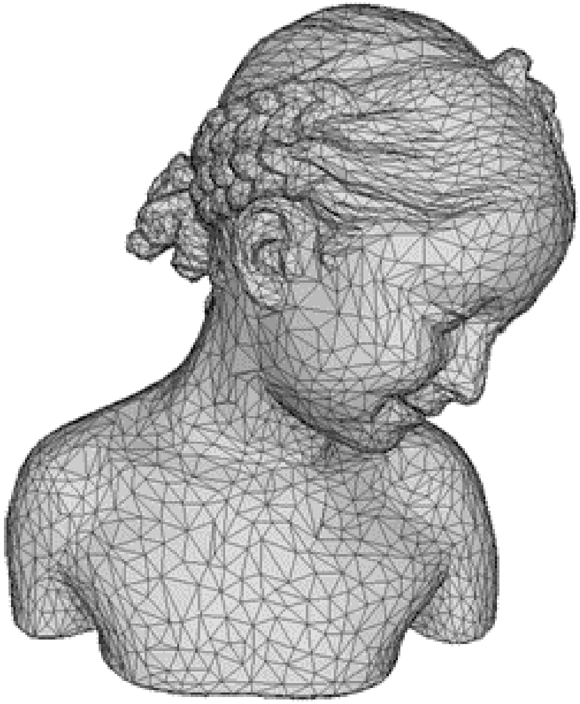
- **Overview (world transformation)**
- Transformation
- OpenGL implementation

Recap: Describing Geometry

- Geometry of an object is defined by specifying the coordinates of the **vertices** and **their adjacencies**



12 triangles



10K triangles

Describing Scenes

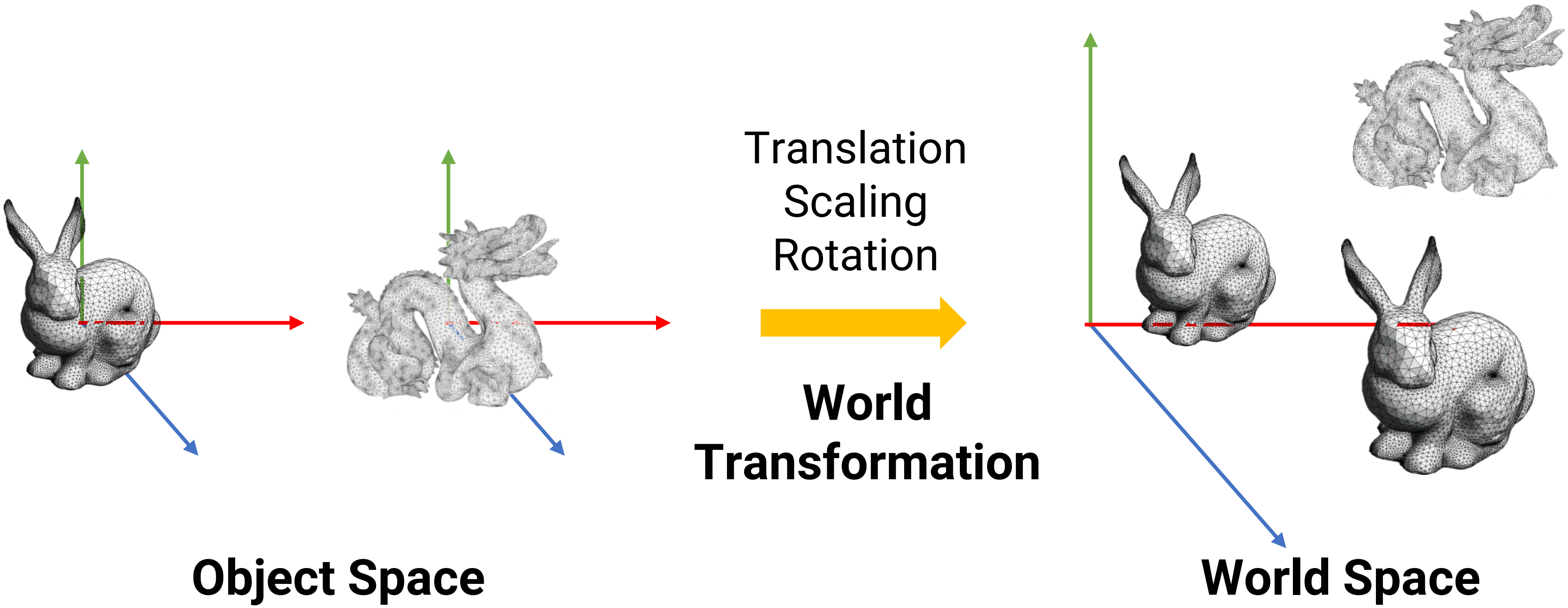
- A virtual scene usually consists of lots of objects

Object Space and World Space

- Objects are defined in **object space** **individually**

Object Space and World Space (cont.)

- Objects are defined in **object space individually**
- When building a scene, each object is transformed to a **global** and **unique** space called **world space**
- The transform is called **world transform**



World Space and World Coordinate (cont.)

- Advantages of using “transformation”
  - **Reuse model:** design a model and use it in several scenes
  - **Memory saving:** store a 4x4 matrix instead of duplicating the entire models

Outline

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- OpenGL implementation

2D Transformations

- 2D transformation of a point can be represented by the multiplication of a **column vector (point)** and a **transformation matrix**

transform matrix

$$\boxed{p'} = \boxed{M}\boxed{p}$$

new point original point

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}}_{p'} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{p}$$

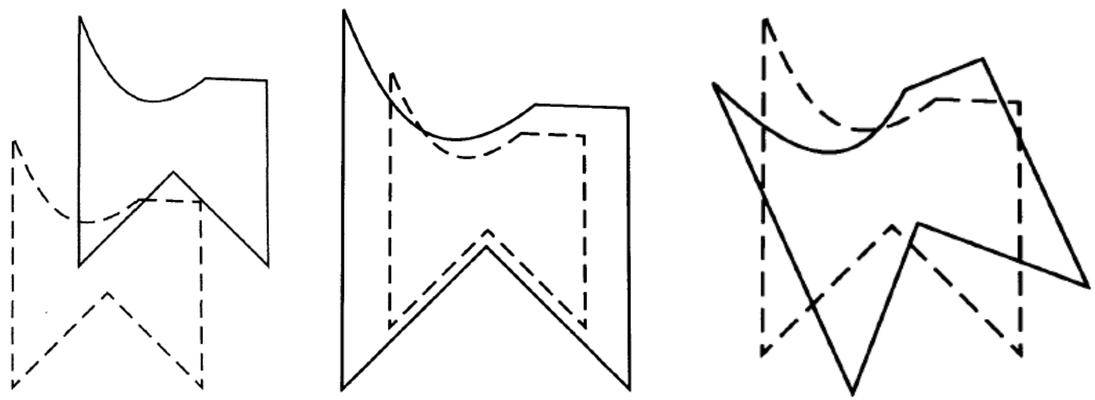
$$x' = ax + by + c$$

$$y' = dx + ey + f$$

- Common transformations include **translation**, **scaling**, and **rotation**

Common Transformations

- Translation
- Scaling
- Rotation



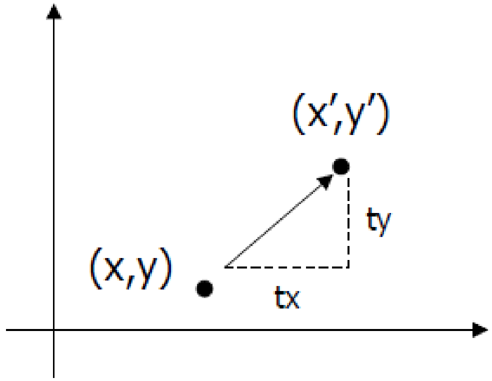
translation scaling rotation

- We will start by introducing 2D transformations and then extend to the 3D cases

2D Translation

- Given a point **p(x, y)** and a **translation offset T(t_x, t_y)**, the new point **p'(x', y')** after translation is **p' = p + T**

$$x' = x + t_x$$
$$y' = y + t_y$$



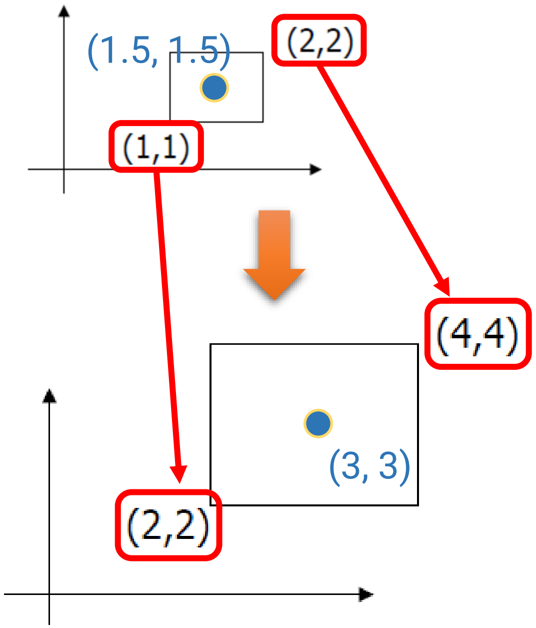
- **Can be represented as** Matrix-vector multiplication

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Scaling

- Given a point **p(x, y)** and a scaling factor **S(s_x, s_y)**, the new point **p'(x', y')** after scaling is **p' = S p**

$$x' = x * s_x$$
$$y' = y * s_y$$

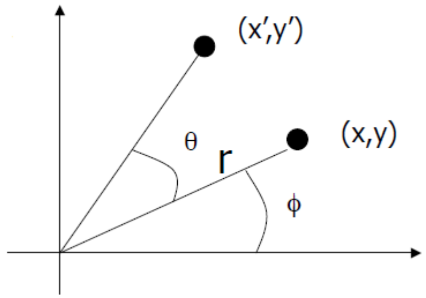


- Matrix-vector multiplication

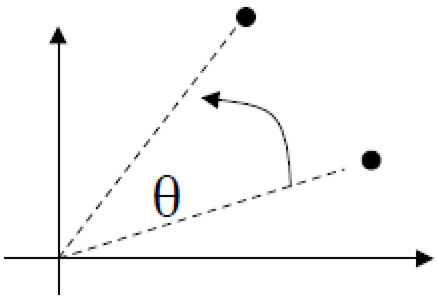
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Rotation

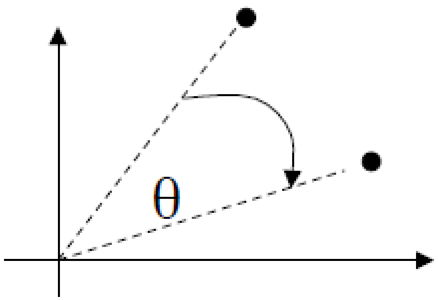
- Given a point **p(x, y)**, rotate it with respect to the **origin** by **θ** and get the new point **p'(x', y')** after rotation



- Firstly we define



θ > 0: rotate counterclockwise



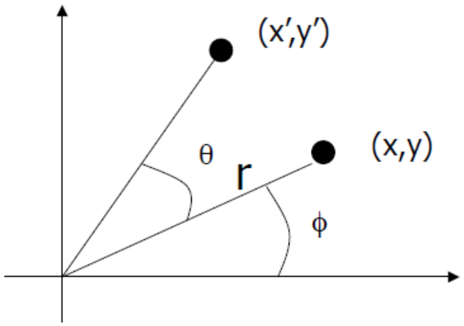
θ < 0: rotate clockwise

2D Rotation (cont.)

- Given a point **p(x, y)**, rotate it with respect to the **origin** by **θ** and get the new point **p'(x', y')** after rotation

$$x = r\cos(\phi) \qquad y = r\sin(\phi)$$

$$x' = r\cos(\phi + \theta) \qquad y' = r\sin(\phi + \theta)$$



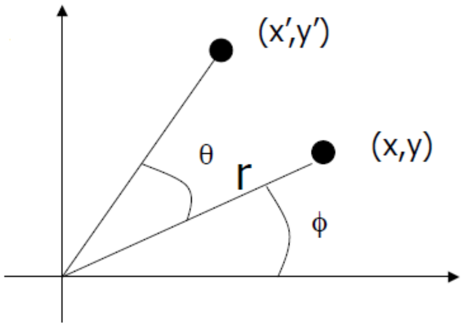
$$\begin{aligned} x' &= r\cos(\phi + \theta) \\ &= r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta) \\ &= x\cos(\theta) - y\sin(\theta) \end{aligned}$$

$$\begin{aligned} y' &= r\sin(\phi + \theta) \\ &= r\sin(\phi)\cos(\theta) + r\cos(\phi)\sin(\theta) \\ &= y\cos(\theta) + x\sin(\theta) \end{aligned}$$

2D Rotation (cont.)

- Given a point **p(x, y)**, rotate it with respect to the **origin** by **θ** and get the new point **p'(x', y')** after rotation

$$\begin{aligned} x' &= r\cos(\phi+\theta) \\ &= x\cos(\theta) - y\sin(\theta) \\ y' &= r\sin(\phi+\theta) \\ &= y\cos(\theta) + x\sin(\theta) \end{aligned}$$



- Matrix-vector multiplication

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Translation, Scaling, and Rotation

- Translation $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
- Scaling $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
- Rotation $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
- Using a 3x3 matrix allows us to perform all transformations using matrix/vector multiplications
  - We can also **pre-multiply (concatenate)** all the matrices

Homogeneous Coordinate

- We call the (x, y, 1) representation the **homogeneous coordinate** for a point (x, y)

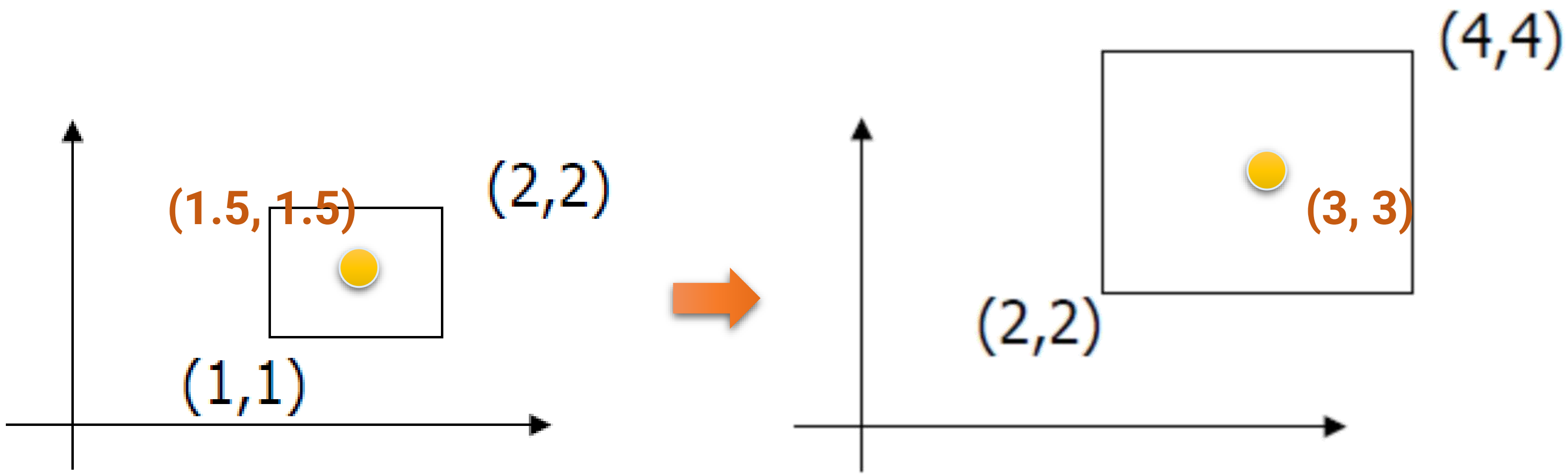
$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- If w is not equal to 1, to make the transformed coordinate also homogeneous, we need to divide the x and y components by w

$$x' = x'/w \qquad y' = y'/w \qquad w = 1$$

Revisit 2D Scaling

- The standard scaling matrix will only anchor at (0, 0)
  - Otherwise, the object center got shifted



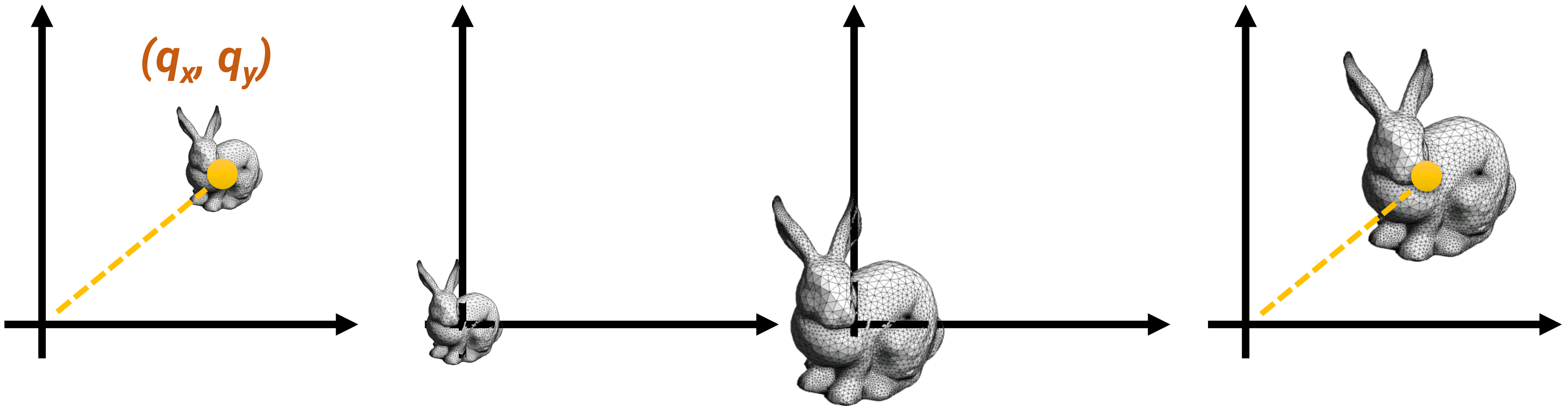
- What if we want the object to be scaled w.r.t its center?

Revisit 2D Scaling (cont.)

- Scaling about an arbitrary pivot point **Q(q_x, q_y)**
  - Translate the objects so that Q will coincide with the origin: **T($-q_x$, $-q_y$)**
  - Scale the object: **S(s_x, s_y)**
  - Translate the object back: **T(q_x, q_y)**

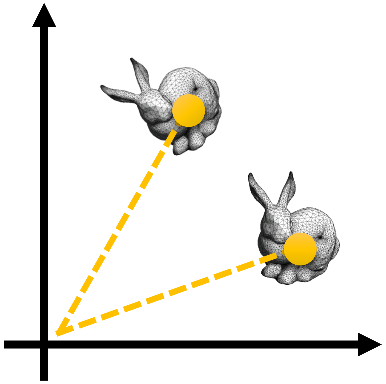
Concatenation of matrices

- The final scaling matrix can be written as $\boxed{T(q)S(s)T(-q)}$



Revisit 2D Rotation

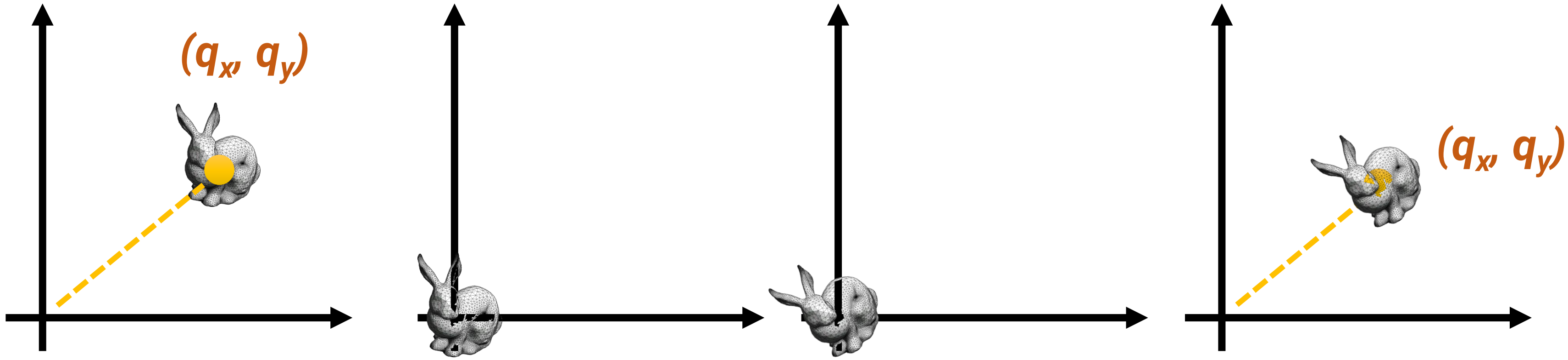
- The standard rotation matrix is used to rotate about the origin (0, 0)



- What if we want the object to be rotated w.r.t a specific pivot?

Revisit 2D Rotation (cont.)

- Rotate about an arbitrary pivot point **Q(q_x, q_y)** by **θ**
  - Translate the objects so that Q will coincide with the origin: **T($-q_x$, $-q_y$)**
  - Rotate the object: **R(θ)**
  - Translate the object back: **T(q_x, q_y)**
- The final rotation matrix can be written as $\boxed{T(q)R(\theta)T(-q)}$



Translation (3D) and Scaling (3D)

- A 3D transformation is represented as a **4x4 matrix**, with **homogeneous coordinate**

translation

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

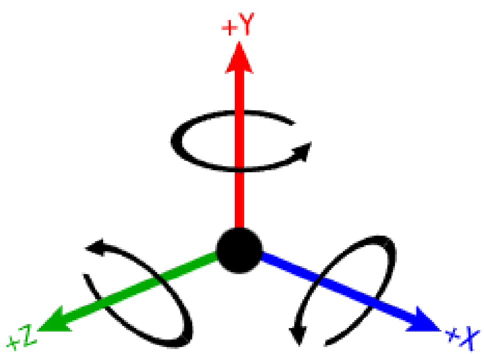
scaling

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2D 3D

Rotation (3D)

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



2D

rotation w.r.t x-axis

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rotation w.r.t y-axis

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rotation w.r.t z-axis

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D

Outline

- Overview (world transformation)
- Transformation
- **OpenGL implementation**

Goals

- Learn how to build the transformation matrices
- Learn how to code with GLM matrices
- Learn how to concatenate the transformation

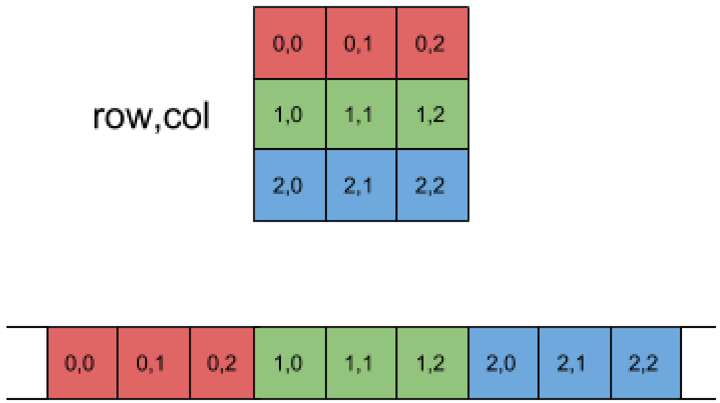
GLM Matrix

- GLM provides several classes to support matrices with different rows and columns
  - Square matrix
    - glm::mat2 (equals to glm::mat2x2)
    - glm::mat3 (equals to glm::mat3x3)
    - glm::mat4 (equals to glm::mat4x4)
  - Non-square matrix
    - glm::mat***m***x***n*** (***m*** and ***n*** are in the range from 2 to 4)

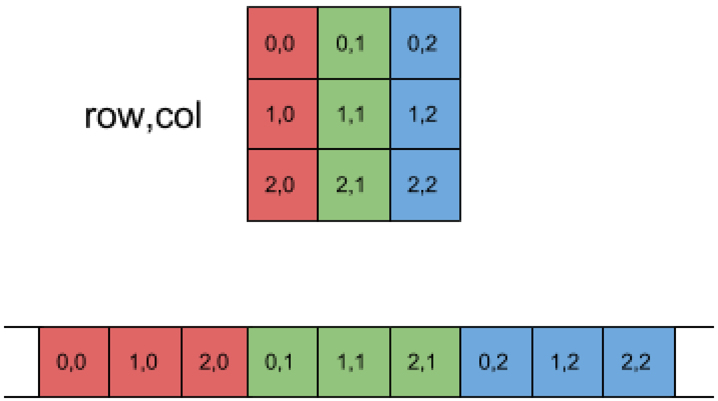
- Declare a **zero** 4x4 matrix: glm::mat4x4(**0.0f**);
- Declare an **identity** 4x4 matrix: glm::mat4x4(**1.0f**);

Matrix Representation: Column/Row Major

- A 2-dimensional matrix can be accessed by either column-major or row-major



row-major



column-major

- By default, OpenGL (and thus GLM) supplies matrix data in **column-major**

Translation Matrix

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

base matrix

- glm::mat4x4 translate(const glm::mat4x4& m ,
  const glm::vec3& v)

returned
translation matrix

translation vector

```
glm::mat4x4 gT = glm::translate(glm::mat4x4(1.0f),
                                glm::vec3(0.1f, 0.2f, 0.3f));
```

Translation Matrix (cont.)

- If you print the matrix produced by glm::translate, you will get the following result

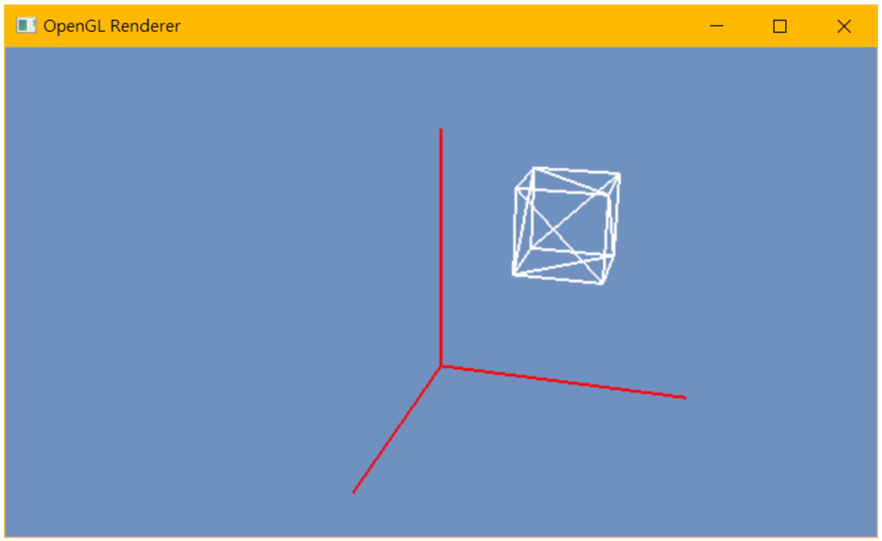
```
GLM's Translation:
    1      0      0      0
    0      1      0      0
    0      0      1      0
  0.1    0.2    0.3      1
```

Why? OpenGL and GLM use column-major representation!

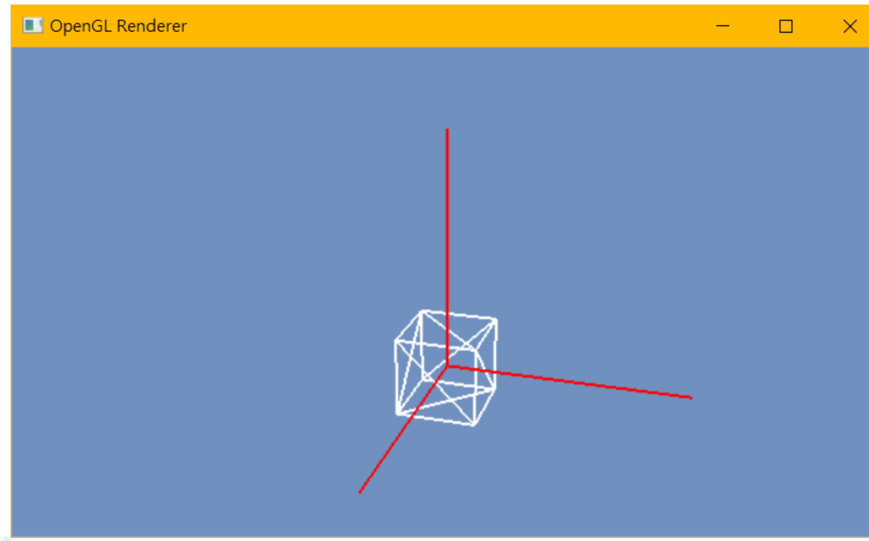
- If you want to build the matrix on your own, remember to transpose the matrix

```
void BuildTranslationMatrix(glm::mat4x4& T, const glm::vec3& tr)
{
    T[0][0] = 1.0f;  T[0][1] = 0.0f;  T[0][2] = 0.0f;  T[0][3] = 0.0f;
    T[1][0] = 0.0f;  T[1][1] = 1.0f;  T[1][2] = 0.0f;  T[1][3] = 0.0f;
    T[2][0] = 0.0f;  T[2][1] = 0.0f;  T[2][2] = 1.0f;  T[2][3] = 0.0f;
    T[3][0] = tr.x;  T[3][1] = tr.y;  T[3][2] = tr.z;  T[3][3] = 1.0f;
}
```

Translation Matrix (cont.)



cube center (1.5, 2.0, 0.0)



cube center (0.0, 0.0, 0.0)

apply a translation of (-1.5, -2.0, 0.0)

Scaling Matrix

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

base matrix

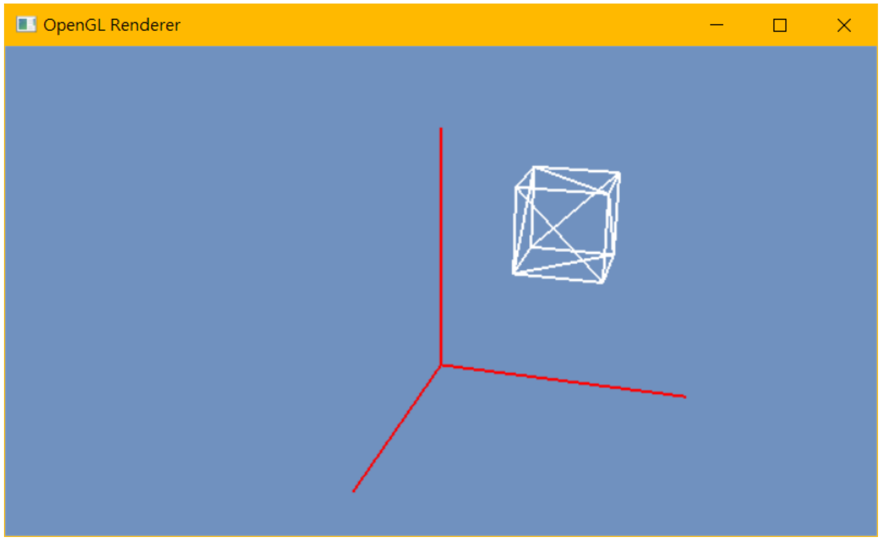
- glm::mat4x4 scale(const glm::mat4x4& m ,
  const glm::vec3& v)

returned
scaling matrix

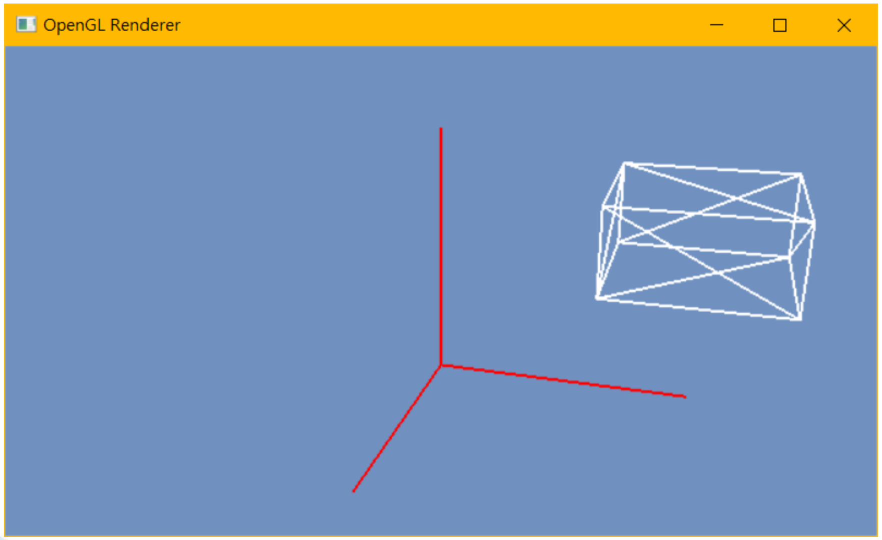
scaling vector

```
glm::mat4x4 gS = glm::scale(glm::mat4x4(1.0f),
                            glm::vec3(0.5f, 0.4f, 0.2f));
```

Translation Matrix (cont.)



cube center (1.5, 2.0, 0.0)



cube center (3.0, 2.0, 0.0)

apply a scaling of (2.0, 1.0, 2.0)

Rotation Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

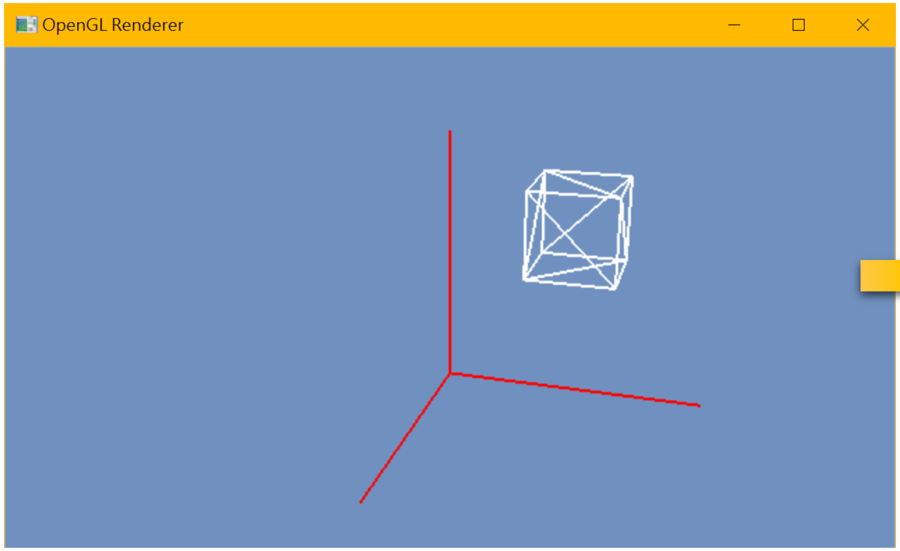
rotation w.r.t x-axis | rotation w.r.t y-axis | rotation w.r.t z-axis

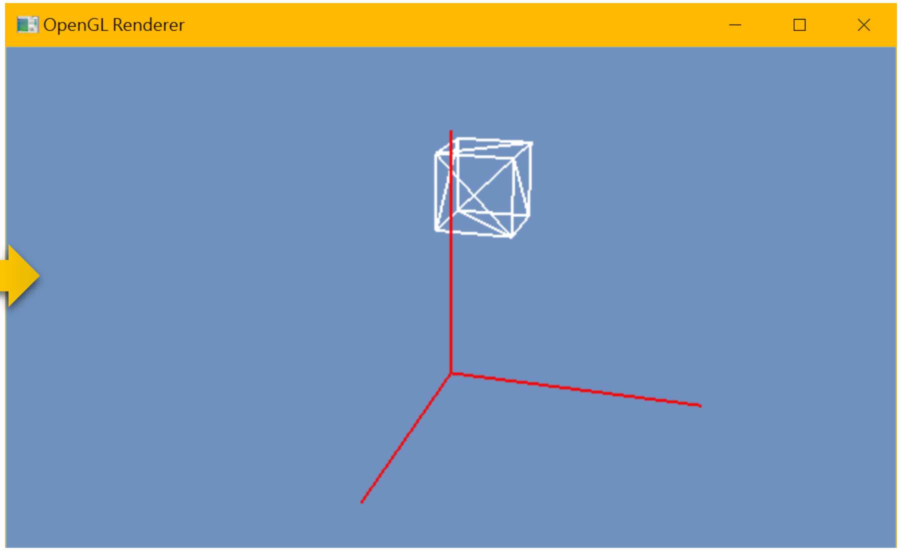
- glm::mat4x4 rotate(const glm::mat4x4& m , base matrix
  const float angle, rotate amount in **radian**
  const glm::vec3& axis) rotate axis

returned scaling matrix

```
glm::mat4x4 gRx = glm::rotate(glm::mat4x4(1.0f), glm::radians(30.0f), glm::vec3(1, 0, 0));
glm::mat4x4 gRy = glm::rotate(glm::mat4x4(1.0f), glm::radians(45.0f), glm::vec3(0, 1, 0));
glm::mat4x4 gRz = glm::rotate(glm::mat4x4(1.0f), glm::radians(60.0f), glm::vec3(0, 0, 1));
```

3D Rotating in Place (cont.)



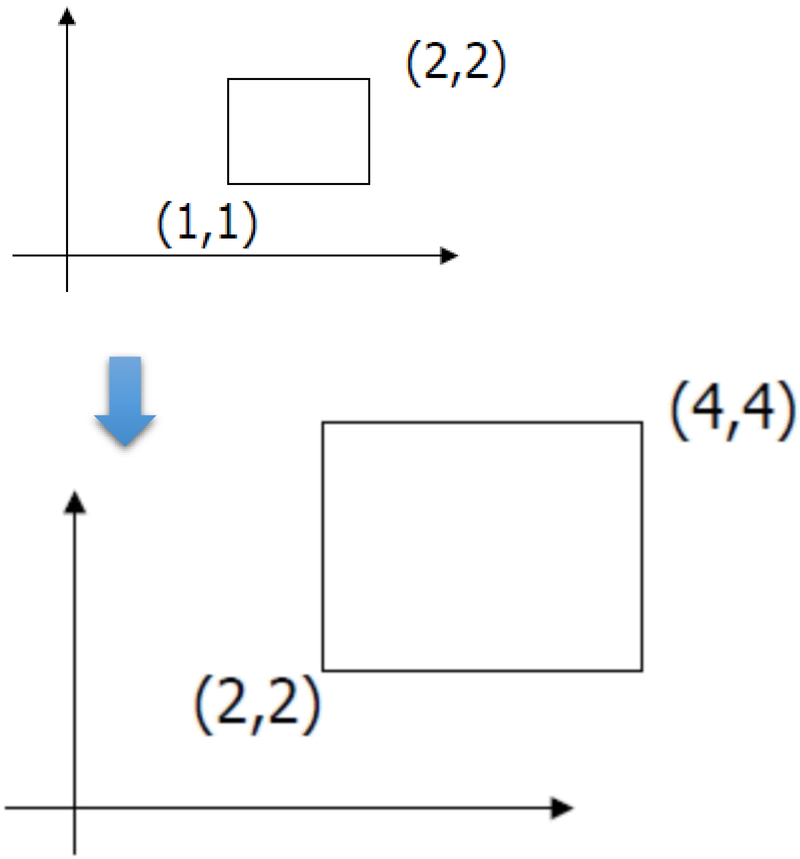


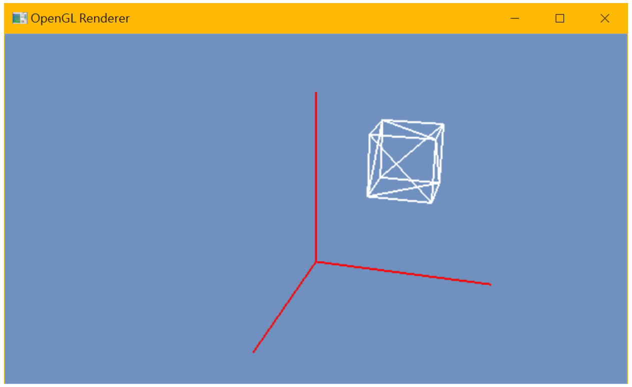

```
glm::mat4x4 RY = glm::rotate(glm::mat4x4(1.0f), glm::radians(90.f), glm::vec3(0, 1, 0));
worldMatrix = RY;
```

rotate w.r.t the global Y axis

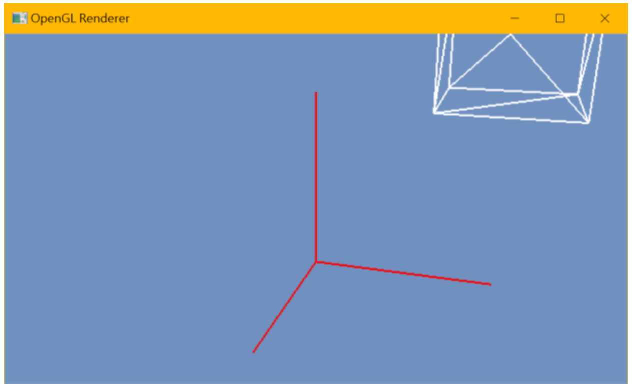
3D Scaling in Place

- The standard scaling matrix will only anchor at (0, 0)





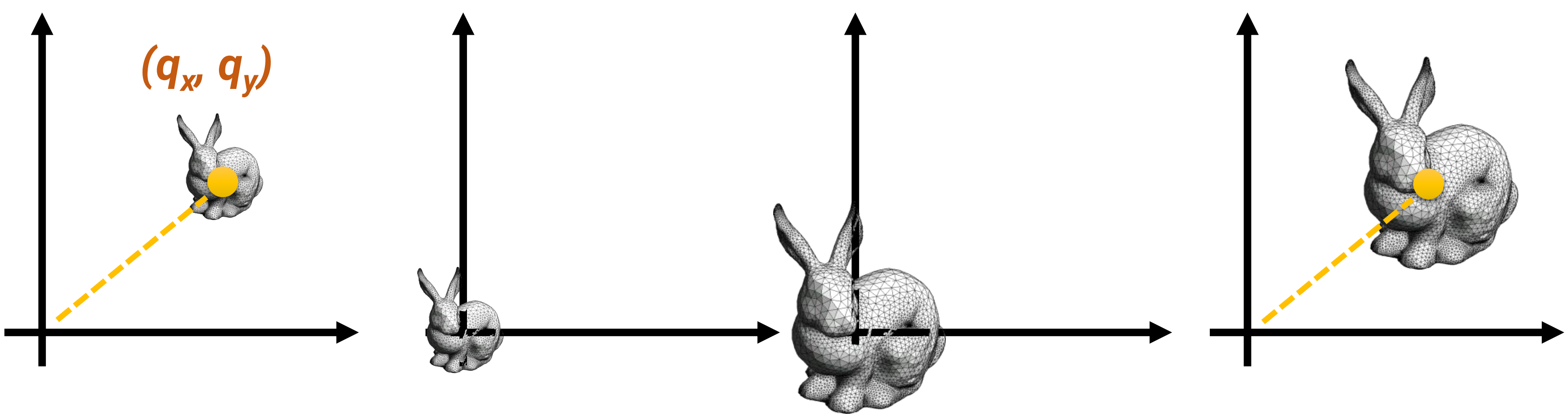
cube center at (1.5, 2.0, 0.0)



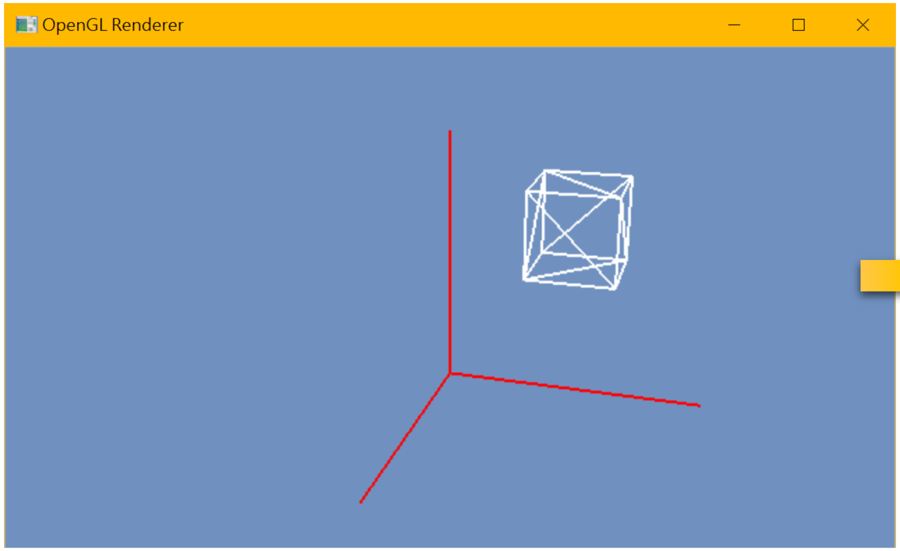
cube center at (3.0, 4.0, 0.0)

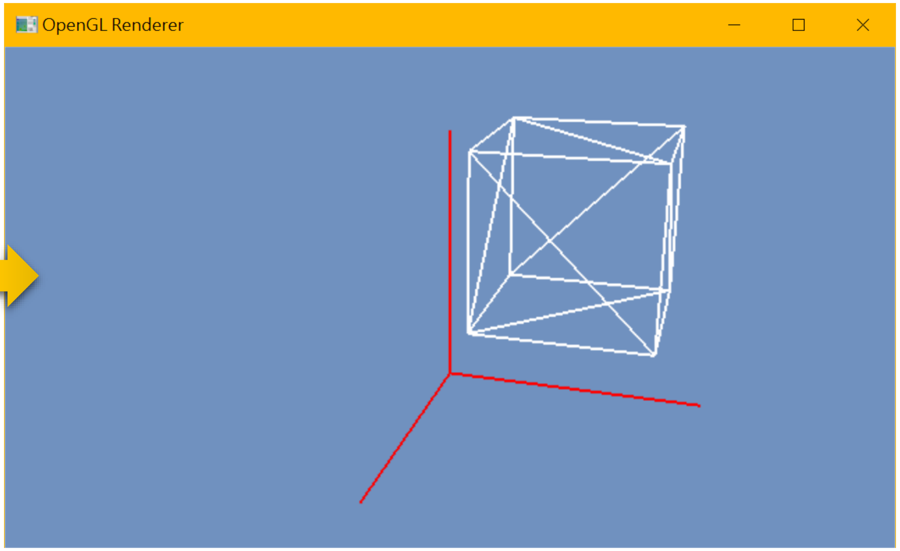
3D Scaling in Place (cont.)

- Scaling about an arbitrary pivot point $Q(q_x, q_y)$
  - Translate the objects so that Q will coincide with the origin: **$T(-q_x, -q_y)$**
  - Scale the object: **$S(s_x, s_y)$**
  - Translate the object back: **$T(q_x, q_y)$**
- The final scaling matrix can be written as $T(q)S(s)T(-q)$



3D Scaling in Place (cont.)

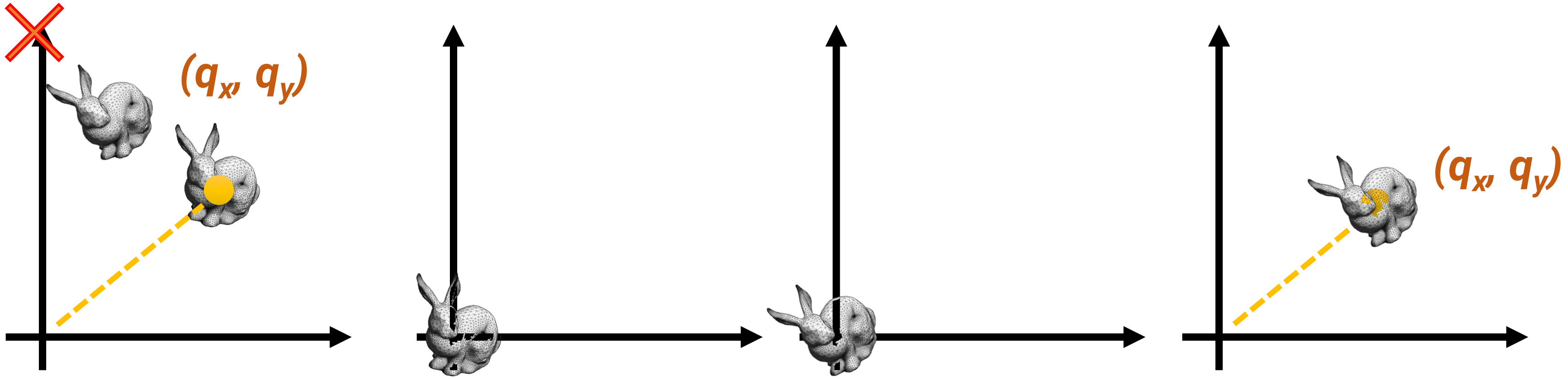




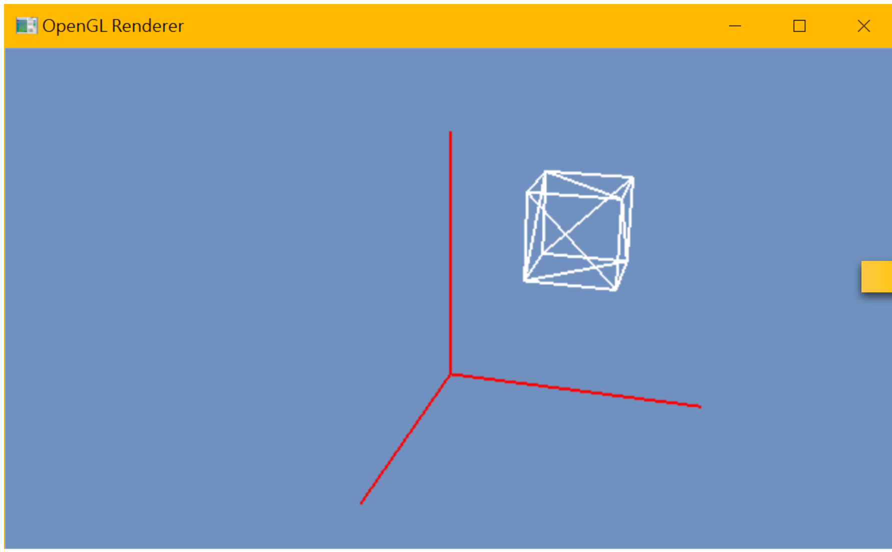

```
glm::mat4x4 T1 = glm::translate(glm::mat4x4(1.0f), glm::vec3(-1.5f, -2.0f, 0.0f));
glm::mat4x4 S = glm::scale(glm::mat4x4(1.0f), glm::vec3(2.0f, 2.0f, 2.0f));
glm::mat4x4 T2 = glm::translate(glm::mat4x4(1.0f), glm::vec3( 1.5f,  2.0f, 0.0f));
worldMatrix = T2 * S * T1;
```

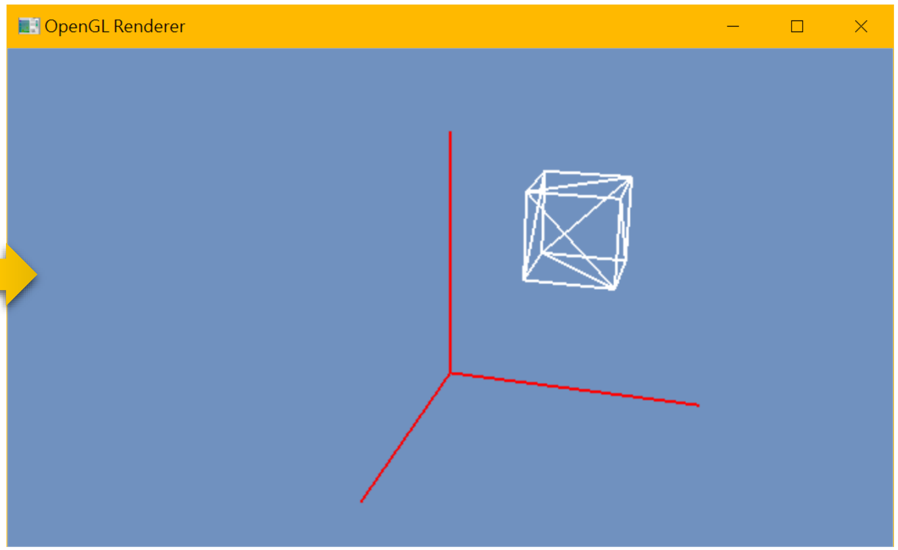
3D Rotating in Place (cont.)

- The standard rotation matrix rotates about an **axis**
- Rotate about an arbitrary pivot point ***Q(q_x, q_y)*** by **θ**
  - Translate the objects so that Q will coincide with the origin: **T(-q_x, -q_y)**
  - Rotate the object: **R(θ)**
  - Translate the object back: **T(q_x, q_y)**
- The final rotation matrix can be written as $\boxed{T(q)R(\theta)T(-q)}$



3D Rotating in Place (cont.)



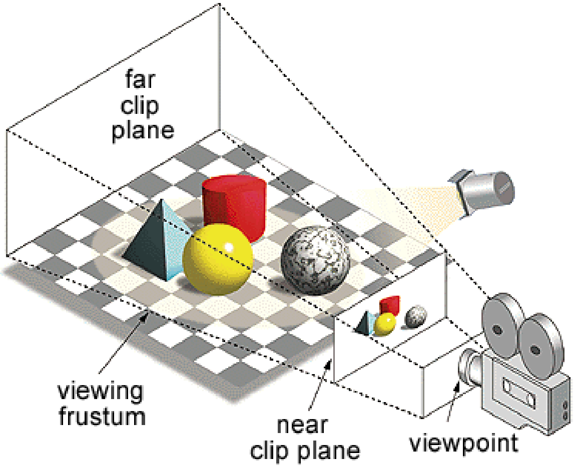



```
glm::mat4x4 T1 = glm::translate(glm::mat4x4(1.0f), glm::vec3(-1.5f, -2.0f, 0.0f));
glm::mat4x4 RY = glm::rotate(glm::mat4x4(1.0f), glm::radians(90.f), glm::vec3(0, 1, 0));
glm::mat4x4 T2 = glm::translate(glm::mat4x4(1.0f), glm::vec3( 1.5f,  2.0f, 0.0f));
worldMatrix = T2 * RY * T1;
```

rotate in place!

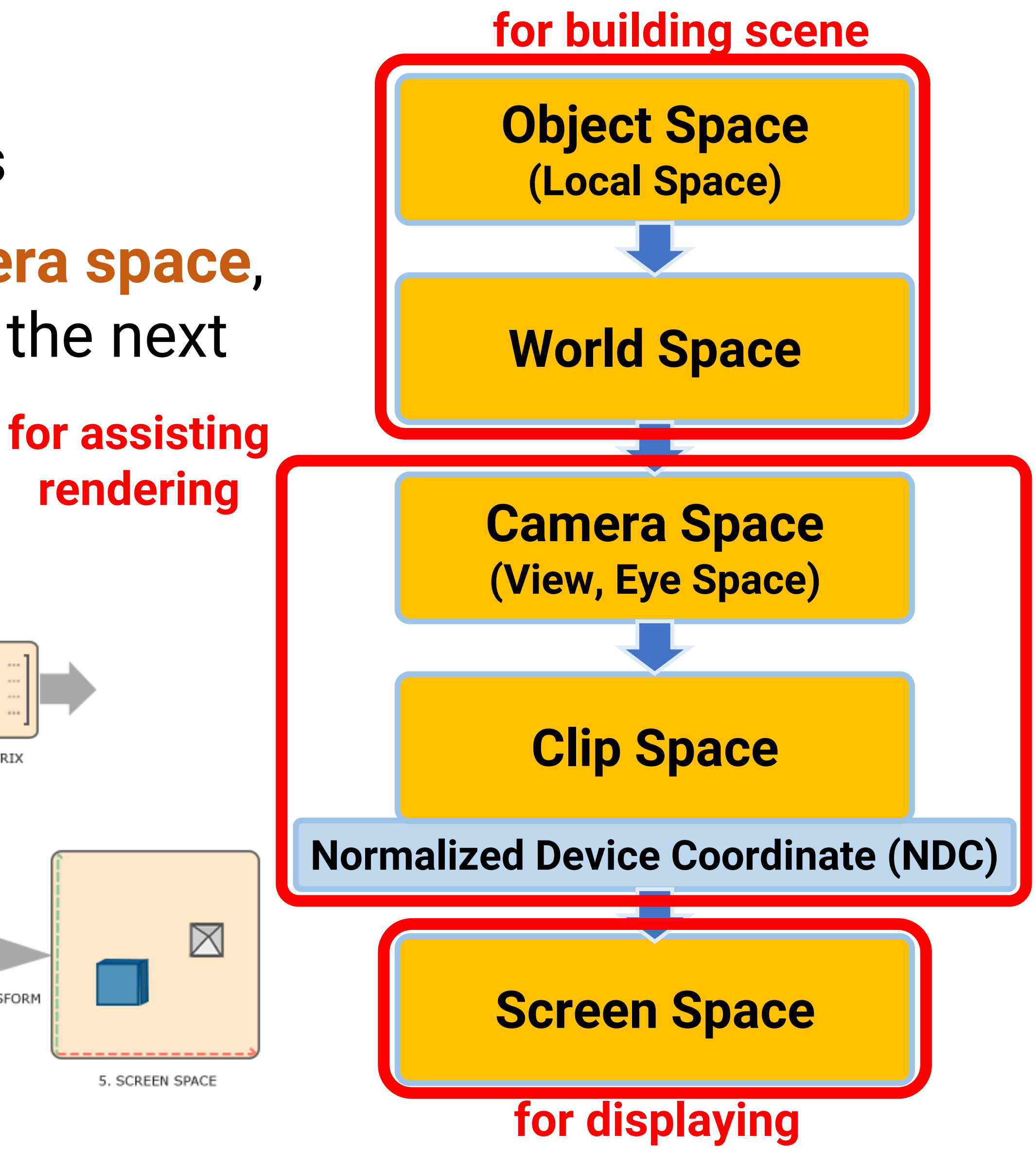
Where is the Camera and Projection?

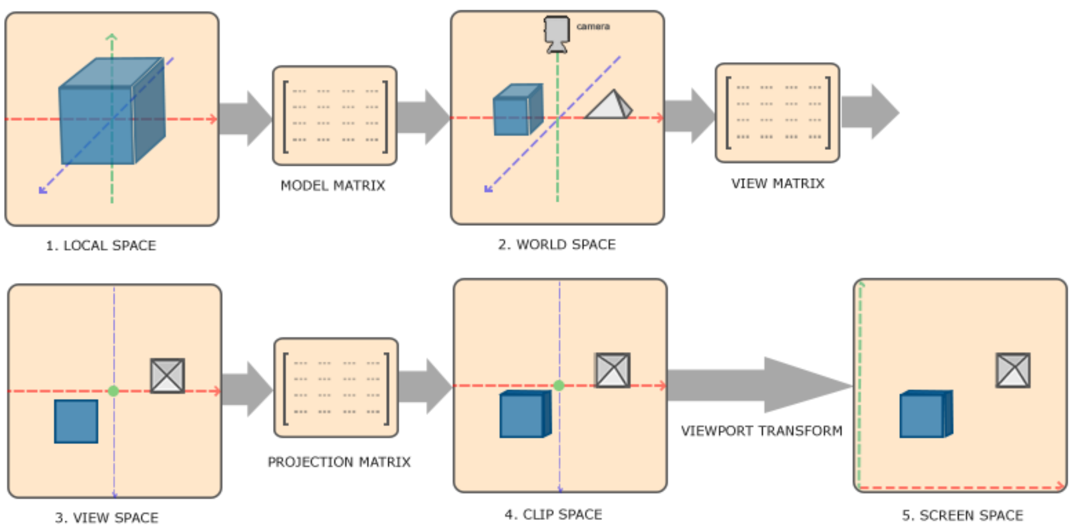
- The typical flow of bringing a 3D point to the 2D screen involves the **camera projection**
- For now, we specify neither the camera nor the projection, so you can consider that we set the **“projected”** positions of the vertices directly
- In the next topic (camera), we will go through the full transformation



Spoiler

- There are other spaces
- We will introduce **camera space**, **clip space**, and **NDC** in the next slides






END OF PRESENTATION
ANY QUESTIONS?